**Plan**

Introduction to
Impartial Combinatorial Games

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**The Beer Can Game**

Players alternate
placing beer cans on
a circular table.
Once placed a can
cannot move. The
first one who
cannot put a can on
the table loses.

What’s the winning
strategy?

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**A Take-Away Game**

Two Players: 1 and 2

A move consists of removing one,
two, or three chips from the pile

Players alternate moves, with
Player 1 starting

Player that removes the last
chip wins

Which player would you rather be?

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**Tangent: Boromean Rings**

Challenge for next time: Generalize to n rings.
Make one of paper and bring it to class.

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**Mathematical Games I**

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Try Small Examples!

If there are 1, 2, or 3 only, player who moves next wins

If there are 4 chips left, player who moves next must leave 1, 2 or 3 chips, and his opponent will win

With 5, 6 or 7 chips left, the player who moves next can win by leaving 4 chips

0, 4, 8, 12, 16, … are target positions; if a player moves to that position, they can win the game

Therefore, with 21 chips, Player 1 can win!

What if the last player to move loses?

If there is 1 chip, the player who moves next loses

If there are 2, 3, or 4 chips left, the player who moves next can win by leaving only 1

In this case, 1, 5, 9, 13, … are a win for the second player

Combinatorial Games

• A set of positions (position = state of the game)
• Two players (know the state)
• Rules specify for each player and for each position which moves to other positions are legal moves
• The players alternate moving
• A terminal position in one in which there are no moves
• The game ends when a player has no moves
• The game must end in a finite number of moves
• (No draws!)

Normal Versus Misère

Normal Play Rule: The last player to move wins

Misère Play Rule: The last player to move loses

A Terminal Position is one where the player has no moves.

What is Omitted

No randomness
• (This rules out Backgammon)
No hidden state
• (This rules out Battleship)
No draws
• (This rules out Chess)

However, Go, Hex and many other games do fit.
Impartial Versus Partizan

A combinatorial game is **impartial** if the same set of moves is available to both players in any position. Example: the take-away game.

A combinatorial game is **partizan** if the move sets may differ for the two players. Example: chess*.

* Make chess a combinatorial by awarding a draw to black.

P-Positions and N-Positions

**For impartial normal games**

P-Position: Positions that are winning for the Previous player (the player who just moved) (Sometimes called LOSING positions)

N-Position: Positions that are winning for the Next player (the player who is about to move) (Sometimes called WINNING positions)

0, 4, 8, 12, 16, … are P-positions; if a player moves to that position, they can win the game

21 chips is an N-position

What's a P-Position?

“Positions that are winning for the Previous player (the player who just moved)”

That means:

For any move that N makes

There exists a move for P such that

For any move that N makes

There exists a move for P such that

There exists a move for P such that

There are no possible moves for N

P-positions and N-positions can be defined recursively by the following:

1. All terminal positions are P-positions (normal winning rule)
2. A position where all moves give N-positions is an P-position
3. A position where at least 1 move gives a P-position is an N-position.
Theorem: Every position in any combinatorial game is game either a P or an N position.

Proof: Immediate from the labeling algorithm on the previous slide, and the fact that the game must end. (It's induction.)

Chomp!

Two-player game, where each move consists of taking a square and removing it and all squares to the right and above. BUT -- You cannot move to (1,1)

Show That This is a P-position

Show That This is an N-position

N-Positions!

P-position!

Let's Play! I'm player 1

No matter what you do, I can mirror it!
Mirroring is an extremely important strategy in combinatorial games!

**Theorem:** A square starting position of Chomp is an N-position (Player 1 can win)

**Proof:**
The winning strategy for player 1 is to chomp on $(2,2)$, leaving only an “L” shaped position.

Then, for any move that Player 2 takes, Player 1 can simply mirror it on the flip side of the “L”

**Theorem:** Every rectangle of area $>1$ is an N-position

**Proof:**
Consider this position:

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+---+---+---+
|   |   |   |
|   |   |   |
|   |   |   |
```

This is either a P or an N-position. If it’s a P-position, then the original rectangle was N. If it’s an N-position, then there exists a move from it to a P-position $X$.

But by the geometry of the situation, $X$ is also available as a move from the starting rectangle. It follows that the original rectangle is an N-position. QED

Notice that this is a non-constructive proof. We’ve shown that there exists a winning move from a rectangle, but we have not found the move.

The Game of Nim

Two players take turns moving

Winner is the last player to remove chips

A move consists of selecting a pile and removing one or more chips from it.

(In one move, you cannot remove chips from more than one pile.)

Analyzing Simple Positions

We use $(x,y,z)$ to denote this position

$(0,0,0)$ is a: P-position

One-Pile Nim

What happens in positions of the form $(x,0,0)$? (with $x>0$)

The first player can just take the entire pile, so $(x,0,0)$ is an N-position
Two-Pile Nim

P-positions are those for which the two piles have an equal number of chips. If it is the opponent’s turn to move from such a position, he must change to a position in which the two piles have different number of chips. From a position with an unequal number of chips, you can easily go to one with an equal number of chips (perhaps the terminal position). (Mirroring again.)

Nim-Sum

The nim-sum of two non-negative integers is their addition without carry in base 2. We will use $\oplus$ to denote the nim-sum.

$2 \oplus 3 = (10)_2 \oplus (11)_2 = (01)_2 = 1$

$5 \oplus 3 = (101)_2 \oplus (011)_2 = (110)_2 = 6$

$7 \oplus 4 = (111)_2 \oplus (100)_2 = (011)_2 = 3$

$\oplus$ is associative: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

$\oplus$ is commutative: $a \oplus b = b \oplus a$

For any non-negative integer $x$,

$$x \oplus x = 0$$

Cancellation Property Holds

If $x \oplus y = x \oplus z$

Then $x \oplus x \oplus y = x \oplus x \oplus z$

So $y = z$

Bouton’s Theorem: A position $(x,y,z)$ in Nim is a P-position if and only if $x \oplus y \oplus z = 0$

Proof:

Let $Z$ denote the set of Nim positions with nim-sum zero

Let $NZ$ denote the set of Nim positions with non-zero nim-sum

We prove the theorem by proving that $Z$ and $NZ$ satisfy the three conditions of P-positions and N-positions

(1) All terminal positions are in $Z$

The only terminal position is $(0,0,0)$

(2) From each position in $NZ$, there is a move to a position in $Z$

\[
\begin{array}{c c}
001010001 & 001010001 \\
100010111 & 100010111 \\
\oplus 111010000 & \oplus 111010000 \\
011010110 & 000000000
\end{array}
\]

Look at leftmost column with an odd # of 1s

Rig any of the numbers with a 1 in that column so that everything adds up to zero
(3) Every move from a position in Z is to a position in NZ

If (x,y,z) is in Z, and x is changed to x' ≠ x, then we cannot have

\[ x \oplus y \oplus z = 0 = x' \oplus y \oplus z \]

Because then x = x' QED