Great Theoretical Ideas In Computer Science

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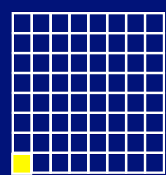
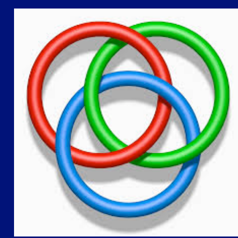
CS 15-251 Spring 2014

Lecture 110 Feb 4, 2014 Carnegie Mellon University

Mathematical Games I



Tangent: Boromean Rings



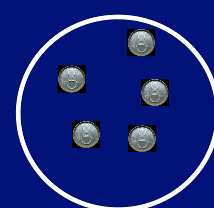
Challenge for next time: Generalize to n rings.
Make one of paper and bring it to class.

Plan

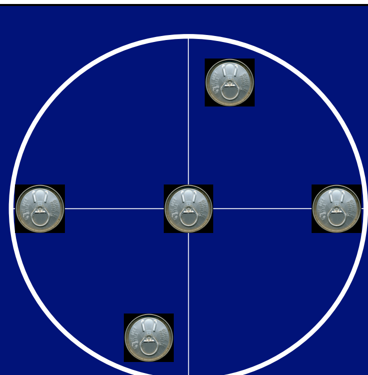
Introduction to
Impartial Combinatorial Games

The Beer Can Game

Players alternate placing beer cans on a circular table. Once placed a can cannot move. The first one who cannot put a can on the table loses.



What's the winning strategy?



A Take-Away Game

21 chips

Two Players: 1 and 2

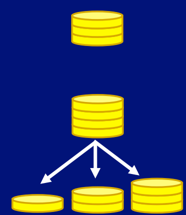
A move consists of removing one, two, or three chips from the pile

Players alternate moves, with Player 1 starting

Player that removes the last chip wins

Which player would you rather be?

Try Small Examples!



If there are 1, 2, or 3 only, player who moves next wins

If there are 4 chips left, player who moves next must leave 1, 2 or 3 chips, and his opponent will win

With 5, 6 or 7 chips left, the player who moves next can win by leaving 4 chips

0, 4, 8, 12, 16, … are target positions; if a player moves to that position, they can win the game

Therefore, with 21 chips, Player 1 can win!

What if the last player to move loses?

If there is 1 chip, the player who moves next loses

If there are 2,3, or 4 chips left, the player who moves next can win by leaving only 1

In this case, 1, 5, 9, 13, … are a win for the second player

Combinatorial Games

- A set of positions (position = state of the game)
- Two players (know the state)
- Rules specify for each player and for each position which moves to other positions are legal moves
- The players alternate moving
- A terminal position in one in which there are no moves
- The game ends when a player has no moves
- The game must end in a finite number of moves
- (No draws!)

Normal Versus Misère

Normal Play Rule: The last player to move wins

Misère Play Rule: The last player to move loses

A Terminal Position is one where the player has no moves.

What is Omitted

No randomness

(This rules out Backgammon)

No hidden state

(This rules out Battleship)

No draws

(This rules out Chess)

However, Go, Hex and many other games do fit.

Impartial Versus Partizan

A combinatorial game is impartial if the same set of moves is available to both players in any position. Example: the take-away game.

A combinatorial game is partizan if the move sets may differ for the two players. Example: chess*.

* Make chess a combinatorial by awarding a draw to black.

Impartial Versus Partizan

In this class we'll study impartial games. Partizan games will not be discussed (except on this slide) They have a deep and beautiful theory too!

See "Winning Ways" by Berelekamp, Guy and Conway

And "Surreal Numbers" by Knuth

Conway developed an alternative definition of numbers based on the theory of partizan games. See "On Numbers and Games".

P-Positions and N-Positions For impartial normal games

P-Position: Positions that are winning for the Previous player (the player who just moved) (Sometimes called LOSING positions)

N-Position: Positions that are winning for the Next player (the player who is about to move) (Sometimes called WINNING positions)

21 chips

0, 4, 8, 12, 16, … are P-positions; if a player moves to that position, they can win the game

21 chips is an N-position

What's a P-Position?

"Positions that are winning for the Previous player (the player who just moved)"

That means:

For any move that N makes

There exists a move for P such that

For any move that N makes

There exists a move for P such that

⋮

There exists a move for P such that

There are no possible moves for N

P-positions and N-positions can be defined recursively by the following:

(1) All terminal positions are P-positions (normal winning rule)

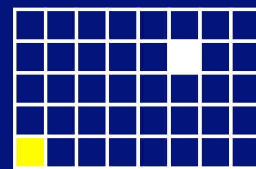
(2) A position where all moves give N-positions is an P-position

(3) A position where at least 1 move gives a P-position is an N-position.

Theorem: Every position in any combinatorial game is game either a P or an N position.

Proof: Immediate from the labeling algorithm on the previous slide, and the fact that the game must end. (It's induction.)

Chomp!



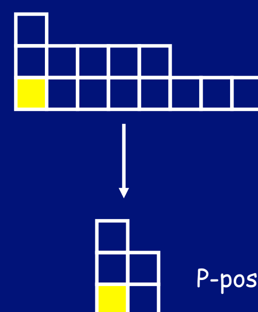
Two-player game, where each move consists of taking a square and removing it and all squares to the right and above.

BUT -- You cannot move to (1,1)

Show That This is a P-position

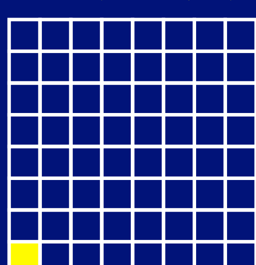
N-Positions!

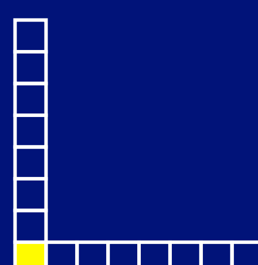
Show That This is an N-position



P-position!

Let's Play! I'm player 1





No matter what you do, I can mirror it!

Mirroring is an extremely important strategy in combinatorial games!

Theorem: A square starting position of Chomp is an N-position (Player 1 can win)

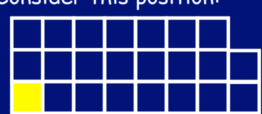
Proof:

The winning strategy for player 1 is to chomp on (2,2), leaving only an “L” shaped position

Then, for any move that Player 2 takes, Player 1 can simply mirror it on the flip side of the “L”

Theorem: Every rectangle of area>1 is a N-position

Proof: Consider this position:



This is either a P or an N-position. If it’s a P-position, then the original rectangle was N. If it’s an N-position, then there exists a move from it to a P-position X.

But by the geometry of the situation, X is also available as a move from the starting rectangle. It follows that the original rectangle is an N-position. **QED**

Notice that this is a non-constructive proof. We’ve shown that there exists a winning move from a rectangle, but we have not found the move.

The Game of Nim

Two players take turns moving

Winner is the last player to remove chips

A move consists of selecting a pile and removing one or more chips from it.

(In one move, you cannot remove chips from more than one pile.)

Analyzing Simple Positions

We use (x,y,z) to denote this position

(0,0,0) is a: P-position

One-Pile Nim

What happens in positions of the form (x,0,0)? (with x>0)

The first player can just take the entire pile, so (x,0,0) is an N-position

Two-Pile Nim

P-positions are those for which the two piles have an equal number of chips.

If it is the opponent's turn to move from such a position, he must change to a position in which the two piles have different number of chips.

From a position with an unequal number of chips, you can easily go to one with an equal number of chips (perhaps the terminal position). (Mirroring again.)

Nim-Sum

The nim-sum of two non-negative integers is their addition without carry in base 2.

We will use $\oplus$ to denote the nim-sum

$$2 \oplus 3 = (10)_2 \oplus (11)_2 = (01)_2 = 1$$

$$5 \oplus 3 = (101)_2 \oplus (011)_2 = (110)_2 = 6$$

$$7 \oplus 4 = (111)_2 \oplus (100)_2 = (011)_2 = 3$$

$\oplus$ is associative: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

$\oplus$ is commutative: $a \oplus b = b \oplus a$

For any non-negative integer x,

$$x \oplus x = 0$$

Cancellation Property Holds

If $x \oplus y = x \oplus z$

Then $x \oplus x \oplus y = x \oplus x \oplus z$

So $y = z$

Bouton's Theorem: A position (x,y,z) in Nim is a P-position if and only if $x \oplus y \oplus z = 0$

Proof:

Let Z denote the set of Nim positions with nim-sum zero

Let NZ denote the set of Nim positions with non-zero nim-sum

We prove the theorem by proving that Z and NZ satisfy the three conditions of P-positions and N-positions

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(1) All terminal positions are in Z

The only terminal position is (0,0,0)

(2) From each position in NZ, there is a move to a position in Z

	001010001	⇒	001010001
	100010111		100010111
$\oplus$	111010000	$\oplus$	101000110
	010010110		000000000

Look at leftmost column with an odd # of 1s

Rig any of the numbers with a 1 in that column so that everything adds up to zero

(3) Every move from a position in Z is to a position in NZ

If (x,y,z) is in Z, and x is changed to $x' \neq x$, then we cannot have

$$x \oplus y \oplus z = 0 = x' \oplus y \oplus z$$

Because then $x = x'$ QED

Study Bee

- Combinatorial games
- Impartial versus Partizan
- Normal Versus Misère
- P-positions versus N-positions
- Mirroring
- Chomp
- Nim
- Nim-sum
- Bouton's Theorem